Cosmological perturbation theory using the FFTLog formalism to massive neutrino cosmologies

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Abstract

These notes aim to calculate via FFTLog formalism the one-loop redshift space galaxy power spectrum in cosmologies in the presence of massive neutrinos. We compare the FFTLog & Direct computations and present a performance test for our public Python code `FOLPS`ν (`https://github.com/henoriega/FOLPS-nu`), which computes the multipoles of the redshift space power spectrum in a fraction of second for massive neutrinos cosmologies as well as for the Einstein-de Sitter (EdS) case.

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1 General framework: `fk`-kernels

It is important to develop consistent theories and accurate methods to account for massive neutrinos because ongoing and forthcoming galaxy surveys such as the Dark Energy Spectroscopic Instrument (DESI[1]), Euclid[2], and the Legacy Survey of Space and Time (LSST[3]) will become increasingly accurate, opening the possibility of measuring the absolute scale of neutrino masses and their mass hierarchy in the coming years. Furthermore, proper modeling of massive neutrinos will provide information about the structure formation and also help to reduce systematic errors in future analyses.

Massive neutrinos introduce an additional scale that causes the linear growth function D_+ and the logarithmic growth rate f to become scale- and time-dependent

$$f(k,t) \equiv \frac{d\ln D_+(k,t)}{d\ln a(t)}. \tag{1.1}$$

In the large scale limit, massive neutrinos behave like CDM. This implies that, on large scales, the logarithmic growth rate in $\nu\Lambda$CDM cosmologies reduces to the usual form in ΛCDM cosmology, $f_0(t) \equiv f(k \to 0, t)$. A direct consequence of working in the presence of massive neutrinos is that the velocity field acquires an extra scale-dependent contribution in linear and non-linear orders. In particular, at linear order, the density contrast δ_{cb} and the divergence of the velocity field θ_{cb} are non-local related by[4]

$$\theta^{(1)}_{cb}(\mathbf{k},t) = \frac{f(k,t)}{f_0(t)}\delta^{(1)}_{cb}(\mathbf{k},t), \tag{1.2}$$

resulting the same at large scales, but suppressed by a factor $f(k)/f_0$ on smaller scales than the characteristic free-streaming scale introduced by massive neutrinos. In general, the non-linear order contributions of the density contrast and velocity divergence are given by[5]

$$\delta^{(n)}_{cb}(\mathbf{k},t) = \int_{\mathbf{k}_{1\cdots n}=\mathbf{k}} F_n(\mathbf{k}_1,\cdots,\mathbf{k}_n;t)\,\delta^{(1)}_{cb}(\mathbf{k}_1,t)\cdots\delta^{(1)}_{cb}(\mathbf{k}_n,t), \tag{1.3}$$

$$\theta^{(n)}_{cb}(\mathbf{k},t) = \int_{\mathbf{k}_{1\cdots n}=\mathbf{k}} G_n(\mathbf{k}_1,\cdots,\mathbf{k}_n;t)\,\delta^{(1)}_{cb}(\mathbf{k}_1,t)\cdots\delta^{(1)}_{cb}(\mathbf{k}_n,t), \tag{1.4}$$

but when considering massive neutrinos, the kernels F_n and G_n are modified due to the free-streaming scale through f and via some functions that must be obtained from second-order differential equations. For example, at linear order in massive neutrino cosmologies we have

$$F_1(\mathbf{k},t) = 1, \qquad \text{and} \qquad G_1(\mathbf{k},t) = \frac{f(k,t)}{f_0(t)}. \tag{1.5}$$

Finding the higher-order kernels is more demanding compared to those in Einstein-de Sitter (EdS) cosmology and represents the result of a lot of work. In ref. [1] the authors developed a Lagrangian Perturbation Theory (LPT) framework to study the clustering of CDM-baryons in the presence of massive neutrinos, finding a self-consistent and well-behaved theory. Then, in ref. [2] the authors mapped the LPT kernels to the Standard Perturbation Theory (SPT) framework. Here, we explicitly write the kernels up to second-order

$$F_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{1}{2} + \frac{3}{14}\mathcal{A} + \left(\frac{1}{2} - \frac{3}{14}\mathcal{B}\right)\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2} + \frac{\mathbf{k}_1\cdot\mathbf{k}_2}{2k_1k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right), \tag{1.6}$$

[1] desi.lbl.gov/
[2] sci.esa.int/web/euclid
[3] lsst.org/
[4] cb refers to CDM-baryon fluid.
[5] We use the shorthand notations

$$\int_{\mathbf{k}_{1\cdots n}=\mathbf{k}} \equiv \int \frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}(2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}_{1\cdots n}), \quad \mathbf{k}_{1\cdots n} \equiv \mathbf{k}_1 + \cdots + \mathbf{k}_n.$$

$$G_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{3\mathcal{A}\left(f(k_1)+f(k_2)\right)+3\dot{\mathcal{A}}/H}{14f_0} + \frac{\mathbf{k}_1\cdot\mathbf{k}_2}{2k_1k_2}\left(\frac{f(k_2)}{f_0}\frac{k_2}{k_1}+\frac{f(k_1)}{f_0}\frac{k_1}{k_2}\right) + \left(\frac{f(k_1)+f(k_2)}{2f_0} - \frac{3\mathcal{B}\left(f(k_1)+f(k_2)\right)+3\dot{\mathcal{B}}/H}{14f_0}\right)\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2}, \tag{1.7}$$

where $\mathcal{A},\mathcal{B}(\mathbf{k}_1,\mathbf{k}_2,t)$ are scale- and time-dependent functions, which are solutions to second-order differential equations. Then, the pertubative kernels F_n and G_n, recieve two additional contributions: one that includes the logarithmic growth rate $f(k,t)$, which is scale-dependent because of the neutrino free-streaming, and scale- and time-dependent functions which arise because $f^2=\Omega_m$ does not hold in massive neutrinos cosmologies. However, when the conditions $\mathcal{A},\mathcal{B}=\mathcal{A}^{M_\nu=0}=1$ and $f(k)=f_0$ are satisfied, the usual EdS kernels are recovered.

In the general case of massive neutrinos cosmologies, the calculation of the loop corrections is computationally slow since at each step of the integration we have to solve differential equations to find the values of $\mathcal{A},\mathcal{B}$, as a consequence, this procedure inhibits employing efficient sampling algorithms to estimate cosmological parameters. However, in ref. [2] the authors identified that considering only the growth rate functions within the perturbative kernels, while keeping the other pieces equal to their EdS values, is a good approximation to the full case (within 0.3 % for $k<0.5\,h\,\mathrm{Mpc}^{-1}$), due to the dominant contribution of massive neutrinos to the loop corrections comes from the growth rate factors rather than the computationally costly functions. Hence, it is convenient to define the `fk`-kernels as

$$F_n^{\mathtt{fk}}, G_n^{\mathtt{fk}}(\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_n) = F_n, G_n(\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_n)\Big|_{\mathcal{A}=\mathcal{B}=\mathcal{A}^{M_\nu=0}}. \tag{1.8}$$

From now on we will work with these kernels, which have two important advantages for efficient evaluation: there is no need to solve differential equations and one can exploit FFTLog methods to speed up even more the loop computations.

2 FFTLog formalism

The one-loop power spectrum involves the computation of several integrals of the form (3.3) − (3.5). In order to simplify and speed up loop calculations, we adapt a tool that decomposes the linear power spectrum into the sum of complex power laws, allowing us to find analytical solutions for the loop integrals. This approach is known as the Fast Fourier Transform in Log-k (FFTLog) formalism [3–6],

$$\bar{P}_L(k) = \sum_{m=-N/2}^{N/2} c_m k^{\nu+i\eta_m}, \tag{2.1}$$

where we have split the discrete approximation to the linear power spectrum in $[k_{\min},k_{\max}]$ logarithmic spaced with N sampling points. The Fourier coefficients c_m and exponents η_m are given by

$$c_m = \frac{W_m}{N}k_{\min}^{-(\nu+i\eta_m)}\sum_{l=0}^{N-1}P_L(k_l)\left(\frac{k_l}{k_{\min}}\right)^{-\nu}e^{-2\pi i m l/N}, \qquad \eta_m = \frac{N-1}{N}\frac{2\pi m}{\log(k_{\max}/k_{\min})}, \tag{2.2}$$

with $W_m=1$ for all m except for $W_{\pm N/2}=1/2$, this last factor prevents endpoints from being counted twice. Notice that $\bar{P}_L(k)$ represents the approximation for the linear power spectrum, while the eq. (2.2) uses the exact linear power spectrum $P_L(k)$. We will continue using the same notation throughout the document. Moreover, the parameter ν is known as bias and in principle can be any real number, but its value can be used to improve the convergence of the loop integrals. All these parameters $\{k_{\min},k_{\max},N,\nu\}$ do not depend on cosmology and we refer to them as the FFTLog parameters. Then the FFTLog formalism decomposes the linear power spectrum into a cosmology-dependent piece c_m and power laws that are independent of cosmology.

On the other hand, the loop integrals involve convolution kernels that can be rewritten as simple multiplications, which together with the FFTLog formalism, allow us to reduce the loop integrals to expressions with the form [7, 8],

$$I(z_1,z_2) \equiv k^{-3+2z_{12}} \int_{\mathbf{p}} \frac{1}{p^{2z_1}|\mathbf{k}-\mathbf{p}|^{2z_2}} = \frac{1}{8\pi^{3/2}} \frac{\Gamma\left(\frac{3}{2}-z_1\right)\Gamma\left(\frac{3}{2}-z_2\right)\Gamma\left(z_{12}-\frac{3}{2}\right)}{\Gamma(z_1)\Gamma(z_2)\Gamma(3-z_{12})}, \tag{2.3}$$

in terms of the Gamma function Γ, where we have employed the shorthand notations $\int_{\mathbf{p}} \equiv (2\pi)^{-3} \int d^3p$ and $z_{12} \equiv z_1 + z_2$ with z_1, z_2 complex numbers.

The above formalism allows transforming the evaluation of the loop integrals into multiplications of cosmology-independent matrices/vectors and cosmology-dependent terms, speeding up the loop calculations compared to the usual way (direct computation). Moreover, as the matrices/vectors do not depend on cosmology, they can be pre-computed and stored, while the cosmology-dependent terms can be found quickly through the usual Fast Fourier Transform (FFT) [6, 9].

On the other hand, notice the function $I(z_1,z_2)$ vanishes if one of the arguments is zero or a negative integer. For example, consider $z_2 = 0$. In this case, eq. (2.3) becomes $\int_{\mathbf{p}} p^{-2z_1} = 0$. On the other hand, it is well known that $\int_{\mathbf{p}} P_L(p)$ diverges. It seems contradictory because the linear power spectrum can be decomposed into power laws, and as a consequence of applying eq. (2.3) we would find zero as the final result. The latter suggests that $I(z_1,z_2)$ gives inappropriate results when the integral is divergent. This is because $I(z_1,z_2)$ calculates only the finite part of the loop integral. Therefore, if the integral we are interested have an ultraviolet (UV) or infrared (IR) divergence, to get the correct answer, one simply has to add the UV/IR contribution by hand [6, 10].

3 One-loop power spectrum

The linear density power spectrum satisfies[6] $P_L(k) \equiv P^L_{cb,\delta\delta}(k) = \langle \delta_{cb}(\mathbf{k})\delta_{cb}(\mathbf{k}')\rangle'$. Then, for massive neutrinos cosmologies (`fk`-kernels), the linear cross-spectrum and the linear velocity power spectrum are

$$P^L_{cb,\delta\theta}(k) = \frac{f(k)}{f_0} P_L(k), \qquad P^L_{cb,\theta\theta}(k) = \left(\frac{f(k)}{f_0}\right)^2 P_L(k). \tag{3.1}$$

In general, the power spectrum up to one-loop has the following form [11],

$$P^{\text{1-loop}}_{ab}(k) = P^L_{ab}(k) + P^{22}_{ab}(k) + P^{13}_{ab}(k), \tag{3.2}$$

where throughout the document a, b stand for the density contrast or the velocity divergence and its combinations. Then, the explicit expressions for the one-loop power spectrum in `fk`-kernels are

$$P^{\text{1-loop}}_{cb,\delta\delta}(k) = P_L(k) + 2\int_{\mathbf{p}} \left[F_2^{\mathtt{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p})\right]^2 P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) + 6\int_{\mathbf{p}} F_3^{\mathtt{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})P_L(k)P_L(p), \tag{3.3}$$

$$\begin{aligned} P^{\text{1-loop}}_{cb,\delta\theta}(k) = P^L_{cb,\delta\theta}(k) &+ 2\int_{\mathbf{p}} F_2^{\mathtt{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p})G_2^{\mathtt{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p})P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\ &+ 3\int_{\mathbf{p}} \left[F_3^{\mathtt{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})\frac{f(k)}{f_0} + G_3^{\mathtt{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})\right] P_L(k)P_L(p), \end{aligned} \tag{3.4}$$

$$P^{\text{1-loop}}_{cb,\theta\theta}(k) = P^L_{cb,\theta\theta}(k) + 2\int_{\mathbf{p}} \left[G_2^{\mathtt{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p})\right]^2 P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) + 6\int_{\mathbf{p}} G_3^{\mathtt{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})\frac{f(k)}{f_0}P_L(k)P_L(p). \tag{3.5}$$

Notice that to compute the one-loop power spectrum, we have to perform the loop integrals (3.3) – (3.5), which are computationally demanding, especially when considering massive neutrino cosmologies. Thus, it is important to develop alternative numerical methods such as the FFTLog to reduce the computational time [3]; otherwise, it would be prohibitive to run efficient sampling algorithms for the estimation of cosmological parameters.

[6] $\langle\cdot\rangle'$ indicates that we have divided by $(2\pi)^3$ times the overall momentum delta function.

UV and IR divergences

Now, let us discuss the convergence properties of eqs. (3.3) − (3.5). Observe the loop integrals are conformed by the convolution of some kernels and linear power spectra. The behavior of fk-kernels under the UV and IR regimes are

$$\text{UV limit } (k \ll p): \quad F_2^{\mathtt{fk}}, G_2^{\mathtt{fk}}(\boldsymbol{p}, \boldsymbol{k}-\boldsymbol{p}) \to \frac{k^2}{p^2}, \qquad F_3^{\mathtt{fk}}, G_3^{\mathtt{fk}}(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) \to \frac{k^2}{p^2}, \tag{3.6}$$

$$\text{IR limit } (p \ll k): \quad F_2^{\mathtt{fk}}, G_2^{\mathtt{fk}}(\mathbf{p}, \mathbf{k}-\mathbf{p}) \to \frac{k}{p}, \qquad F_3^{\mathtt{fk}}, G_3^{\mathtt{fk}}(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) \to \frac{k^2}{p^2}. \tag{3.7}$$

Then, when considering the FFTLog formalism, eq. (2.1), the convergence of the loop integrals is determined by the bias parameter ν. For example, if we take the UV/IR limit and consider $P_L(k) \sim k^\nu$, one can find that the P_{ab}^{22} contributions are UV convergent for $\nu < 1/2$ and IR convergent for $-1 < \nu$. Then for $-1 < \nu < 1/2$ the P_{ab}^{22} integrals are convergent, and consequently, the use of eq. (2.3) return the same results as with the traditional direct computation. Nevertheless, when we choose ν values outside the convergence range, the P_{ab}^{22} integrals become UV or IR divergent. Then, the eq. (2.3) does not guarantee the correct answer because the divergent pieces are not captured by dimensional regularization. Therefore, to obtain the correct answer, the corresponding UV or IR piece must be added to the result obtained through the FFTLog formalism [6, 10].

A similar analysis for contributions P_{ab}^{13} finds these loop integrals are divergent for $\nu > -1$ and $\nu < -1$, i.e. the integrals never converge. Thus, based on the ν value, we have to add the UV or IR divergence, as appropriate. For example, for $\nu > -1$ the loop integrals of P_{ab}^{13} are UV divergent. In this limit the contribution $P_{cb,\delta\theta}^{13}$ turns

$$\begin{aligned}
P_{cb,\delta\theta}^{13,\mathrm{UV}}(k) &= 3P_L(k)\int_0^\infty \frac{dp}{4\pi^2}\, p^2 P_L(p) \int_{-1}^{1} dx \left[F_3^{\mathtt{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})\frac{f(k)}{f_0} + G_3^{\mathtt{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})\right] \\
&= 3P_L(k)\int_0^\infty \frac{dp}{4\pi^2}\, p^2 P_L(p) \int_{-1}^{1} dx \left[\left(\frac{1-32x^2+10x^4}{63}\frac{f(k)}{f_0}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{1-11x^2+10x^4}{63}\frac{f(p)}{f_0}\right)\frac{k^2}{p^2} + \mathcal{O}\left(\frac{k^4}{p^4}\right)\right] \\
&= -3P_L(k)\int_0^\infty \frac{dp}{4\pi^2}\, p^2 P_L(p) \left[\left(\frac{46}{189}\frac{f(k)}{f_0} + \frac{4}{189}\frac{f(p)}{f_0}\right)\frac{k^2}{p^2} + \mathcal{O}\left(\frac{k^4}{p^4}\right)\right].
\end{aligned} \tag{3.8}$$

Then, the leading UV contribution is

$$P_{cb,\delta\theta}^{13,\mathrm{UV}}(k) = -\left(\frac{23}{21}\frac{f(k)}{f_0}\sigma_\Psi^2 + \frac{2}{21}\sigma_v^2\right)k^2 P_L(k), \tag{3.9}$$

with

$$\sigma_\Psi^2 \equiv \frac{1}{6\pi^2}\int_0^\infty dp\, P_L(p), \qquad \sigma_v^2 \equiv \frac{1}{6\pi^2}\int_0^\infty dp\, P_{\delta\theta}^L(p). \tag{3.10}$$

Notice that we have kept only the leading UV contribution, which is valid in the range $-1 < \nu < 1$. For biases $\nu > 1$, one has to consider the subleading UV terms.

Similarly, if we consider biases $\nu < -1$, the IR terms are required. For these biases, the leading IR contribution of $P_{cb,\delta\theta}^{13}$ takes

$$P_{cb,\delta\theta}^{13,\mathrm{IR}}(k) = -P_{\delta\theta}^L(k)k^2\sigma_\Psi^2, \tag{3.11}$$

which holds for $-3 < \nu < -1$. For smaller biases, the subleading IR contributions are necessary.

Returning to P_{ab}^{22}, we find that for the density-velocity spectra, the leading UV and IR contributions are

$$P^{22,\mathrm{UV}}_{cb,\delta\theta}(k) = -\frac{3k^4}{196\pi^2}\int_0^\infty dp\,\frac{f(p)}{f_0}\frac{P_L^2(p)}{p^2}, \qquad (\tfrac{1}{2} < \nu < \tfrac{3}{2}) \tag{3.12}$$

$$P^{22,\mathrm{IR}}_{cb,\delta\theta}(k) = P^L_{cb,\delta\theta}(k)k^2\sigma^2_\Psi, \qquad (-3 < \nu < -1) \tag{3.13}$$

In the table 1 we summarize all the leading UV and IR contributions with the range where they become necessary. Notice that P^{22}_{ab} and P^{13}_{ab} are individually IR divergent for $\nu < -1$. Furthermore, due to the equivalence principle the IR divergences cancel, which implies the loop contribution $P^{\mathrm{loop}}_{ab} \equiv P^{22}_{ab} + P^{13}_{ab}$ become convergent for $-3 < \nu < -1$ [12]. Thus, when considering P^{loop}_{ab} in this range, no IR or UV contributions are needed.

UV and IR divergences		
Contribution	UV div	IR div
P^{22}	$\frac{1}{2} < \nu < \frac{3}{2}$	$-3 < \nu < -1$
	$P^{22,\mathrm{UV}}_{cb,\delta\delta}(k) = \frac{9k^4}{196\pi^2}\int_0^\infty dp\,\frac{P_L^2(p)}{p^2}$	$P^{22,\mathrm{IR}}_{cb,\delta\delta}(k) = P_L(k)k^2\sigma^2_\Psi$
	$P^{22,\mathrm{UV}}_{cb,\delta\theta}(k) = -\frac{3k^4}{196\pi^2}\int_0^\infty dp\,\frac{f(p)}{f_0}\frac{P_L^2(p)}{p^2}$	$P^{22,\mathrm{IR}}_{cb,\delta\theta}(k) = P^L_{cb,\delta\theta}(k)k^2\sigma^2_\Psi$
	$P^{22,\mathrm{UV}}_{cb,\theta\theta}(k) = \frac{k^4}{196\pi^2}\int_0^\infty dp\,\left(\frac{f(p)}{f_0}\right)^2\frac{P_L^2(p)}{p^2}$	$P^{22,\mathrm{IR}}_{cb,\theta\theta}(k) = P^L_{cb,\theta\theta}(k)k^2\sigma^2_\Psi$
P^{13}	$-1 < \nu < 1$	$-3 < \nu < -1$
	$P^{13,\mathrm{UV}}_{cb,\delta\delta}(k) = -\frac{61}{105}P_L(k)k^2\sigma^2_\Psi$	$P^{13,\mathrm{IR}}_{cb,\delta\delta}(k) = -P_L(k)k^2\sigma^2_\Psi$
	$P^{13,\mathrm{UV}}_{cb,\delta\theta}(k) = -\left(\frac{23}{21}\frac{f(k)}{f_0}\sigma^2_\Psi + \frac{2}{21}\sigma^2_v\right)k^2P_L(k)$	$P^{13,\mathrm{IR}}_{cb,\delta\theta}(k) = -P^L_{\delta\theta}(k)k^2\sigma^2_\Psi$
	$P^{13,\mathrm{UV}}_{cb,\theta\theta}(k) = -\left(\frac{169}{105}\frac{f(k)}{f_0}\sigma^2_\Psi + \frac{4}{21}\sigma^2_v\right)\frac{f(k)}{f_0}k^2P_L(k)$	$P^{13,\mathrm{IR}}_{cb,\theta\theta}(k) = -P^L_{\theta\theta}(k)k^2\sigma^2_\Psi$

Table 1: Leading UV/IR contributions and the range of the bias where they are valid.

FFTLog evaluation for the one-loop power spectrum

In this section, we will calculate through the FFTLog formalism the different contributions to the one-loop power spectrum, eqs. (3.3)–(3.5).

Density contributions

We start with $P^{22}_{cb,\delta\delta}(k)$, which is given by

$$P^{22}_{cb,\delta\delta}(k) = 2\int_{\mathbf{p}}\left[F_2^{\mathrm{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p})\right]^2 P_L(p)P_L(|\mathbf{k}-\mathbf{p}|), \tag{3.0.14}$$

with

$$
\begin{aligned}
\left[F_2^{\mathrm{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p})\right]^2 =& \frac{k^8}{196|\mathbf{k}-\mathbf{p}|^4p^4} + \frac{3k^6}{196|\mathbf{k}-\mathbf{p}|^4p^2} + \frac{3k^6}{196|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{29k^4}{392|\mathbf{k}-\mathbf{p}|^2p^2} - \frac{15k^2p^2}{392|\mathbf{k}-\mathbf{p}|^4} \\
&- \frac{15k^2|\mathbf{k}-\mathbf{p}|^2}{392p^4} - \frac{11k^4}{784|\mathbf{k}-\mathbf{p}|^4} + \frac{15k^2}{392|\mathbf{k}-\mathbf{p}|^2} - \frac{11k^4}{784p^4} + \frac{15k^2}{392p^2} + \frac{25p^4}{784|\mathbf{k}-\mathbf{p}|^4} \\
&- \frac{25p^2}{196|\mathbf{k}-\mathbf{p}|^2} - \frac{25|\mathbf{k}-\mathbf{p}|^2}{196p^2} + \frac{25|\mathbf{k}-\mathbf{p}|^4}{784p^4} + \frac{75}{392} \\
\equiv& \sum_{n_1,n_2=-2}^{2} f_{22,\delta\delta}(n_1,n_2)k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2},
\end{aligned} \tag{3.0.15}
$$

and

$$
f_{22,\delta\delta}(n_1,n_2) = \begin{array}{cc}
\begin{array}{ccccc} n_2=-2 & -1 & 0 & 1 & 2 \end{array} & \\
\begin{pmatrix}
\frac{1}{196} & \frac{3}{196} & -\frac{11}{784} & -\frac{15}{392} & \frac{25}{784} \\
\frac{3}{196} & \frac{29}{392} & \frac{15}{392} & -\frac{25}{196} & 0 \\
-\frac{11}{784} & \frac{15}{392} & \frac{75}{392} & 0 & 0 \\
-\frac{15}{392} & -\frac{25}{196} & 0 & 0 & 0 \\
\frac{25}{784} & 0 & 0 & 0 & 0
\end{pmatrix} &
\begin{array}{c} n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}
\end{array}. \tag{3.0.16}
$$

where we have written the expansion at eq. (3.0.15) compactly since all the summands in the kernel above have the form $k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2}$ with $n_1, n_2 \in \{-2,-1,0,1,2\}$.

Using the FFTLog decomposition (2.1), we can write the approximation to $P^{22}_{cb,\delta\delta}(k)$ as follows

$$
\bar{P}^{22}_{cb,\delta\delta}(k) = 2\sum_{m_1,m_2} c_{m_1}c_{m_2}\sum_{n_1,n_2} f_{22,\delta\delta}(n_1,n_2)k^{-2(n_1+n_2)}\int_{\mathbf{p}}\frac{1}{p^{2(\nu_1-n_1)}|\mathbf{k}-\mathbf{p}|^{2(\nu_2-n_2)}}, \tag{3.0.17}
$$

where ν_1 and ν_2 are complex number defined throughout all the document by

$$
\nu_1 \equiv -\frac{1}{2}(\nu + i\eta_{m_1}) \quad \text{and} \quad \nu_2 \equiv -\frac{1}{2}(\nu + i\eta_{m_2}). \tag{3.0.18}
$$

Appealing to the loop integral (2.3), one can write the approximation for $P^{22}_{cb,\delta\delta}(k)$ as

$$
\bar{P}^{22}_{cb,\delta\delta}(k) = k^3\sum_{m_1,m_2} c_{m_1}k^{-2\nu_1}M_{22,\delta\delta}(\nu_1,\nu_2)\,c_{m_2}k^{-2\nu_2}, \tag{3.0.19}
$$

with

$$
\begin{aligned}
M_{22,\delta\delta}(\nu_1,\nu_2) &\equiv 2\sum_{n_1,n_2=-2}^{2} f_{22,\delta\delta}(n_1,n_2)I(\nu_1-n_1,\nu_2-n_2) \\
&= \left[\nu_1\nu_2\left(98\nu_{12}^2 - 14\nu_{12} + 36\right) - 91\nu_{12}^2 + 3\nu_{12} + 58\right] \\
&\quad \times \frac{\left(\frac{3}{2}-\nu_{12}\right)\left(\frac{1}{2}-\nu_{12}\right)}{196\nu_1(1+\nu_1)\left(\frac{1}{2}-\nu_1\right)\nu_2(1+\nu_2)\left(\frac{1}{2}-\nu_2\right)}I(\nu_1,\nu_2),
\end{aligned} \tag{3.0.20}
$$

where $\nu_{12} \equiv \nu_1 + \nu_2$, throughout this document we adopt the notation $\nu_{1\cdots n} \equiv \nu_1 + \cdots + \nu_n$.

Now it is turn to focus our attention to the contribution $P^{13}_{cb,\delta\delta}(k)$, which is given by

$$
P^{13}_{cb,\delta\delta}(k) = 6\int_{\mathbf{p}} F_3^{\mathrm{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})P_L(k)P_L(p), \tag{3.0.21}
$$

where

$$
\begin{aligned}
F_3^{\mathrm{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p}) =& -\frac{k^6}{252|\mathbf{k}-\mathbf{p}|^2p^4} - \frac{k^4}{504|\mathbf{k}-\mathbf{p}|^2p^2} + \frac{19k^2|\mathbf{k}-\mathbf{p}|^2}{168p^4} + \frac{p^4}{72k^2|\mathbf{k}-\mathbf{p}|^2} - \frac{|\mathbf{k}-\mathbf{p}|^4}{18k^2p^2} \\
&+ \frac{|\mathbf{k}-\mathbf{p}|^6}{72k^2p^4} + \frac{5k^2}{168|\mathbf{k}-\mathbf{p}|^2} + \frac{|\mathbf{k}-\mathbf{p}|^2}{12k^2} - \frac{11k^4}{252p^4} - \frac{4k^2}{63p^2} - \frac{p^2}{18k^2} - \frac{19p^2}{504|\mathbf{k}-\mathbf{p}|^2} \\
&+ \frac{61|\mathbf{k}-\mathbf{p}|^2}{504p^2} - \frac{5|\mathbf{k}-\mathbf{p}|^4}{63p^4} - \frac{1}{252} \\
\equiv& \sum_{n_1=-2}^{2}\sum_{n_2=-1}^{3} f_{13,\delta\delta}(n_1,n_2)k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2},
\end{aligned}
\tag{3.0.22}
$$

with

$$
f_{13,\delta\delta}(n_1,n_2) = \begin{array}{c} \begin{array}{ccccc} n_2=-1 & 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -\frac{1}{252} & -\frac{11}{252} & \frac{19}{168} & -\frac{5}{63} & \frac{1}{72} \\ -\frac{1}{504} & -\frac{4}{63} & \frac{61}{504} & -\frac{1}{18} & 0 \\ \frac{5}{168} & -\frac{1}{252} & \frac{1}{12} & 0 & 0 \\ -\frac{19}{504} & -\frac{1}{18} & 0 & 0 & 0 \\ \frac{1}{72} & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} . \tag{3.0.23}
$$

Then, through the eqs. (2.1) and (3.0.22), the approximation for $P^{13}_{cb,\delta\delta}(k)$ can be written as

$$
\bar{P}^{13}_{cb,\delta\delta}(k) = 6P_L(k)\sum_{m_1} c_{m_1} \sum_{n_1,n_2} f_{13,\delta\delta}(n_1,n_2)k^{-2(n_1+n_2)} \int_{\mathbf{p}} \frac{1}{p^{2(\nu_1-n_1)}|\mathbf{k}-\mathbf{p}|^{-2n_2}}, \tag{3.0.24}
$$

Using the eq. (2.3) to solve the integral in the last equation, we find

$$
\bar{P}^{13}_{cb,\delta\delta}(k) = k^3 P_L(k)\sum_{m_1} c_{m_1} k^{-2\nu_1} M_{13,\delta\delta}(\nu_1), \tag{3.0.25}
$$

with

$$
\begin{aligned}
M_{13,\delta\delta}(\nu_1) &\equiv 6\sum_{n_1=-2}^{2}\sum_{n_2=-1}^{3} f_{13,\delta\delta}(n_1,n_2) I(\nu_1-n_1,-n_2) \\
&= 6\sum_{n_1=-2}^{2} f_{13,\delta\delta}(n_1,-1) I(\nu_1-n_1,1).
\end{aligned}
\tag{3.0.26}
$$

In the last equality, we have used that the function $I(z_1,z_2)$ vanishes if one of its arguments is zero or a negative integer. The latter expression can be reduced even more, as follows

$$
M_{13,\delta\delta}(\nu_1) = \frac{1+9\nu_1}{4}\frac{\tan(\nu_1\pi)}{28\pi(\nu_1+1)\nu_1(\nu_1-1)(\nu_1-2)(\nu_1-3)}. \tag{3.0.27}
$$

where we have used the properties of the Gamma function.

Density-velocity contributions

We move our attention to the density-velocity spectra, where

$$
\begin{aligned}
P^{22}_{cb,\delta\theta}(k) &= 2\int_{\mathbf{p}} F_2^{\mathrm{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p}) G_2^{\mathrm{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p}) P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\
&= 2\int_{\mathbf{p}} K^{f_{\mathbf{p}}}_{\delta\theta}(\mathbf{p},\mathbf{k}-\mathbf{p})\frac{f(p)}{f_0}P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) + 2\int_{\mathbf{p}} K^{f_{\mathbf{k}-\mathbf{p}}}_{\delta\theta}(\mathbf{p},\mathbf{k}-\mathbf{p})\frac{f(|\mathbf{k}-\mathbf{p}|)}{f_0}P_L(p)P_L(|\mathbf{k}-\mathbf{p}|).
\end{aligned}
\tag{3.0.28}
$$

In the second equality of eq. (3.0.28) we have used $F_2^{\mathbf{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p})G_2^{\mathbf{fk}}(\mathbf{p},\mathbf{k}-\mathbf{p}) = (f(p)/f_0)\, K_{\delta\theta}^{f_{\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p}) + (f(|\mathbf{k}-\mathbf{p}|/f_0)\, K_{\delta\theta}^{f_{\mathbf{k}-\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p})$, where

$$\begin{aligned} K_{\delta\theta}^{f_{\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p}) =& \frac{k^8}{196|\mathbf{k}-\mathbf{p}|^4p^4} + \frac{3k^6}{196|\mathbf{k}-\mathbf{p}|^4p^2} - \frac{k^6}{392|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{23k^4}{784|\mathbf{k}-\mathbf{p}|^2p^2} - \frac{15k^2p^2}{392|\mathbf{k}-\mathbf{p}|^4} \\ &+ \frac{13k^2|\mathbf{k}-\mathbf{p}|^2}{392p^4} - \frac{11k^4}{784|\mathbf{k}-\mathbf{p}|^4} + \frac{11k^2}{196|\mathbf{k}-\mathbf{p}|^2} - \frac{9k^4}{392p^4} - \frac{5k^2}{98p^2} + \frac{25p^4}{784|\mathbf{k}-\mathbf{p}|^4} \\ &- \frac{65p^2}{784|\mathbf{k}-\mathbf{p}|^2} + \frac{5|\mathbf{k}-\mathbf{p}|^2}{784p^2} - \frac{5|\mathbf{k}-\mathbf{p}|^4}{392p^4} + \frac{45}{784}, \end{aligned} \tag{3.0.29}$$

and

$$\begin{aligned} K_{\delta\theta}^{f_{\mathbf{k}-\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p}) =& \frac{k^8}{196|\mathbf{k}-\mathbf{p}|^4p^4} - \frac{k^6}{392|\mathbf{k}-\mathbf{p}|^4p^2} + \frac{3k^6}{196|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{23k^4}{784|\mathbf{k}-\mathbf{p}|^2p^2} + \frac{13k^2p^2}{392|\mathbf{k}-\mathbf{p}|^4} \\ &- \frac{15k^2|\mathbf{k}-\mathbf{p}|^2}{392p^4} - \frac{9k^4}{392|\mathbf{k}-\mathbf{p}|^4} - \frac{5k^2}{98|\mathbf{k}-\mathbf{p}|^2} - \frac{11k^4}{784p^4} + \frac{11k^2}{196p^2} + \frac{5p^2}{784|\mathbf{k}-\mathbf{p}|^2} \\ &- \frac{5p^4}{392|\mathbf{k}-\mathbf{p}|^4} - \frac{65|\mathbf{k}-\mathbf{p}|^2}{784p^2} + \frac{25|\mathbf{k}-\mathbf{p}|^4}{784p^4} + \frac{45}{784}. \end{aligned} \tag{3.0.30}$$

Notice that under the interchange $\mathbf{p}\to\mathbf{k}-\mathbf{p}$, the above kernels satisfy $K_{\delta\theta}^{f_{\mathbf{k}-\mathbf{p}}}(\mathbf{k}-\mathbf{p},\mathbf{p}) = K_{\delta\theta}^{f_{\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p})$, this fact help to reduce the expression for $P_{cb,\delta\theta}^{22}(k)$ to a single term[7]

$$P_{cb,\delta\theta}^{22}(k) = 4\int_{\mathbf{p}} K_{\delta\theta}^{f_{\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p})P_{cb,\delta\theta}^{L}(p)P_L(|\mathbf{k}-\mathbf{p}|), \tag{3.0.31}$$

where we have used $P_{cb,\delta\theta}^{L}(k) = (f(k)/f_0)\,P_L(k)$. By the same arguments as before, we can rewrite the kernel $K_{\delta\theta}^{f_{\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p})$ as

$$K_{\delta\theta}^{f_{\mathbf{p}}}(\mathbf{p},\mathbf{k}-\mathbf{p}) \equiv \sum_{n_1,n_2=-2}^{2} f_{22,\delta\theta}^{f_{\mathbf{p}}}(n_1,n_2)k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2}, \tag{3.0.32}$$

with

$$f_{22,\delta\theta}^{f_{\mathbf{p}}}(n_1,n_2) = \begin{array}{cc} \begin{matrix} n_2=-2 & -1 & 0 & 1 & 2 \end{matrix} & \\ \begin{pmatrix} \frac{1}{196} & -\frac{1}{392} & -\frac{9}{392} & \frac{13}{392} & -\frac{5}{392} \\ \frac{3}{196} & \frac{23}{784} & -\frac{5}{98} & \frac{5}{784} & 0 \\ -\frac{11}{784} & \frac{11}{196} & \frac{45}{784} & 0 & 0 \\ -\frac{15}{392} & -\frac{65}{784} & 0 & 0 & 0 \\ \frac{25}{784} & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix} \end{array}. \tag{3.0.33}$$

In order to apply the FFTLog method in the eq. (3.0.31) we need a similar expansion to eq. (2.1) for the cross-spectra $P_{cb,\delta\theta}^{L}(k) = (f(k)/f_0)\,P_L(k)$, so we approximate

$$\bar{P}_{cb,\delta\theta}^{L}(k) = \sum_{m=-N/2}^{N/2} c_m^f k^{\nu+i\eta_m}, \tag{3.0.34}$$

where the coefficients c_m^f are computed via the eq. (2.2), but changing $P_L(k)\to(f(k)/f_0)\,P_L(k)$. Then, the approximation of $P_{cb,\delta\theta}^{22}(k)$ can be written as

$$\bar{P}_{cb,\delta\theta}^{22}(k) = 2k^3\sum_{m_1,m_2} c_{m_1}^f k^{-2\nu_1}M_{22,\delta\theta}^{f_{\mathbf{p}}}(\nu_1,\nu_2)c_{m_2}k^{-2\nu_2}, \tag{3.0.35}$$

[7]This is possible because the original form of $P_{cb,\delta\theta}^{22}(k)$ is symmetric under the interchange $\mathbf{p}\to\mathbf{k}-\mathbf{p}$.

with

$$M^{f_{\mathbf{p}}}_{22,\delta\theta}(\nu_1,\nu_2) = \Big[7(7\nu_1+3)\nu_2^2 + (7\nu_1(7\nu_1-1)-38)\nu_2 - 21\nu_1 - 23\Big] \\ \times \frac{(2\nu_{12}-3)(2\nu_{12}-1)}{196\nu_1(\nu_1+1)\nu_2(\nu_2+1)(2\nu_2-1)} I(\nu_1,\nu_2). \tag{3.0.36}$$

On the other hand, the contribution $P^{13}_{cb,\delta\theta}(k)$ has the following form

$$\begin{aligned} P^{13}_{cb,\delta\theta}(k) &= 3\int_{\mathbf{p}} \left[F_3^{\mathbf{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})\frac{f(k)}{f_0} + G_3^{\mathbf{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})\right] P_L(k)P_L(p) \\ &= 3\frac{f(k)}{f_0}P_L(k)\int_{\mathbf{p}} K^{f_{\mathbf{k}}}_{\delta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p})P_L(p) + 3P_L(k)\int_{\mathbf{p}} K^{f_{\mathbf{p}}}_{\delta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p})P^L_{cb,\delta\theta}(p). \end{aligned} \tag{3.0.37}$$

In the last equality we have divided $G_3^{\mathbf{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})$ into two pieces, one of them proportional to $f(p)/f_0$ and the other proportional to $f(k)/f_0$. The latter piece was combined with $F_3(\mathbf{k},-\mathbf{p},\mathbf{p})\,(f(k)/f_0)$, resulting in the new kernels

$$\begin{aligned} K^{f_{\mathbf{k}}}_{\delta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p}) &= -\frac{k^6}{126|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{k^4}{72|\mathbf{k}-\mathbf{p}|^2p^2} + \frac{19k^2|\mathbf{k}-\mathbf{p}|^2}{84p^4} + \frac{5p^4}{504k^2|\mathbf{k}-\mathbf{p}|^2} - \frac{47|\mathbf{k}-\mathbf{p}|^4}{504k^2p^2} \\ &+ \frac{|\mathbf{k}-\mathbf{p}|^6}{36k^2p^4} + \frac{k^2}{168|\mathbf{k}-\mathbf{p}|^2} + \frac{19|\mathbf{k}-\mathbf{p}|^2}{168k^2} - \frac{11k^4}{126p^4} - \frac{73k^2}{504p^2} - \frac{29p^2}{504k^2} - \frac{11p^2}{504|\mathbf{k}-\mathbf{p}|^2} \\ &+ \frac{113|\mathbf{k}-\mathbf{p}|^2}{504p^2} - \frac{10|\mathbf{k}-\mathbf{p}|^4}{63p^4} - \frac{11}{252} \\ &\equiv \sum_{n_1=-2}^{2}\sum_{n_2=-1}^{3} f^{f_{\mathbf{k}}}_{13,\delta\theta}(n_1,n_2)k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2}, \end{aligned} \tag{3.0.38}$$

$$f^{f_{\mathbf{k}}}_{13,\delta\theta}(n_1,n_2) = \begin{array}{cl} \begin{array}{ccccc} n_2=-1 & 0 & 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} -\frac{1}{126} & -\frac{11}{126} & \frac{19}{84} & -\frac{10}{63} & \frac{1}{36} \\ \frac{1}{72} & -\frac{73}{504} & \frac{113}{504} & -\frac{47}{504} & 0 \\ \frac{1}{168} & -\frac{11}{252} & \frac{19}{168} & 0 & 0 \\ -\frac{11}{504} & -\frac{29}{504} & 0 & 0 & 0 \\ \frac{5}{504} & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \end{array}, \tag{3.0.39}$$

and

$$\begin{aligned} K^{f_{\mathbf{p}}}_{\delta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p}) &= -\frac{k^6}{126|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{k^4}{72|\mathbf{k}-\mathbf{p}|^2p^2} + \frac{k^2|\mathbf{k}-\mathbf{p}|^2}{168p^4} + \frac{5p^4}{504k^2|\mathbf{k}-\mathbf{p}|^2} - \frac{5|\mathbf{k}-\mathbf{p}|^4}{126k^2p^2} \\ &+ \frac{5|\mathbf{k}-\mathbf{p}|^6}{504k^2p^4} + \frac{k^2}{168|\mathbf{k}-\mathbf{p}|^2} + \frac{5|\mathbf{k}-\mathbf{p}|^2}{84k^2} + \frac{k^4}{72p^4} + \frac{k^2}{252p^2} - \frac{5p^2}{126k^2} - \frac{11p^2}{504|\mathbf{k}-\mathbf{p}|^2} \\ &+ \frac{11|\mathbf{k}-\mathbf{p}|^2}{504p^2} - \frac{11|\mathbf{k}-\mathbf{p}|^4}{504p^4} + \frac{11}{504} \\ &\equiv \sum_{n_1=-2}^{2}\sum_{n_2=-1}^{3} f^{f_{\mathbf{p}}}_{13,\delta\theta}(n_1,n_2)k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2}, \end{aligned} \tag{3.0.40}$$

$$
f^{f_{\mathbf{p}}}_{13,\delta\theta}(n_1,n_2) = \begin{array}{c} \begin{array}{ccccc} n_2=-1 & 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{ccccc} -\frac{1}{126} & \frac{1}{72} & \frac{1}{168} & -\frac{11}{504} & \frac{5}{504} \\ \frac{1}{72} & \frac{1}{252} & \frac{11}{504} & -\frac{5}{126} & 0 \\ \frac{1}{168} & \frac{11}{504} & \frac{5}{84} & 0 & 0 \\ -\frac{11}{504} & -\frac{5}{126} & 0 & 0 & 0 \\ \frac{5}{504} & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} . \tag{3.0.41}
$$

Then, the approximation for $P^{13}_{cb,\delta\theta}(k)$ takes the form

$$
\bar{P}^{13}_{cb,\delta\theta}(k) = \frac{1}{2}k^3 P_L(k)\left(\frac{f(k)}{f_0}\sum_{m_1} c_{m_1}k^{-2\nu_1}M^{f_{\mathbf{k}}}_{13,\delta\theta}(\nu_1) + \sum_{m_1} c^f_{m_1}k^{-2\nu_1}M^{f_{\mathbf{p}}}_{13,\delta\theta}(\nu_1)\right), \tag{3.0.42}
$$

with

$$
\begin{aligned}
M^{f_{\mathbf{k}}}_{13,\delta\theta}(\nu_1) &\equiv 6\sum_{n_1=-2}^{2}\sum_{n_2=-1}^{3} f^{f_{\mathbf{k}}}_{13,\delta\theta}(n_1,n_2)I\left(\nu_1-n_1,-n_2\right) \\
&= 6\sum_{n_1=-2}^{2} f^{f_{\mathbf{k}}}_{13,\delta\theta}(n_1,-1)I(\nu_1-n_1,1) \\
&= M^{f_{\mathbf{p}}}_{13,\delta\theta}(\nu_1) \\
&= \frac{9\nu_1-7}{4}\,\frac{\tan(\nu_1\pi)}{28\pi(\nu_1+1)\nu_1(\nu_1-1)(\nu_1-2)(\nu_1-3)}.
\end{aligned} \tag{3.0.43}
$$

The equality $M^{f_{\mathbf{k}}}_{13,\delta\theta}(\nu_1) = M^{f_{\mathbf{p}}}_{13,\delta\theta}(\nu_1)$ holds because the first column of eqs. (3.0.39) and (3.0.41) are identical and represent the only nonzero contribution to the vector $M_{13,\delta\theta}$. This allows us to write the approximation for $P^{13}_{cb,\delta\theta}(k)$ as

$$
\bar{P}^{13}_{cb,\delta\theta}(k) = \frac{1}{2}k^3 P_L(k)\sum_{m_1}\left(\frac{f(k)}{f_0}c_{m_1} + c^f_{m_1}\right)k^{-2\nu_1}M^{f_{\mathbf{k}}}_{13,\delta\theta}(\nu_1). \tag{3.0.44}
$$

Velocity contributions

We will now focus on the velocity contribution to the one-loop power spectrum, where

$$
\begin{aligned}
P^{22}_{cb,\theta\theta}(k) &= 2\int_{\mathbf{p}}\left[G^{\mathrm{f}\mathbf{k}}_2(\mathbf{p},\mathbf{k}-\mathbf{p})\right]^2 P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\
&= 2\int_{\mathbf{p}} K^{f_{(\mathbf{k}-\mathbf{p})^2}}_{\theta\theta}(\mathbf{p},\mathbf{k}-\mathbf{p})\frac{f^2(|\mathbf{k}-\mathbf{p}|)}{f_0^2}P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\
&\quad + 2\int_{\mathbf{p}} K^{f_{\mathbf{k}-\mathbf{p}}f_p}_{\theta\theta}(\mathbf{p},\mathbf{k}-\mathbf{p})\frac{f(|\mathbf{k}-\mathbf{p}|)f(p)}{f_0^2}P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\
&\quad + 2\int_{\mathbf{p}} K^{f_{\mathbf{p}^2}}_{\theta\theta}(\mathbf{p},\mathbf{k}-\mathbf{p})\frac{f^2(p)}{f_0^2}P_L(p)P_L(|\mathbf{k}-\mathbf{p}|),
\end{aligned} \tag{3.0.45}
$$

where

$$
\begin{aligned}
K^{f_{(\mathbf{k}-\mathbf{p})^2}}_{\theta\theta}(\mathbf{p},\mathbf{k}-\mathbf{p}) =& \frac{k^8}{196|\mathbf{k}-\mathbf{p}|^4p^4} - \frac{k^6}{49|\mathbf{k}-\mathbf{p}|^4p^2} + \frac{3k^6}{196|\mathbf{k}-\mathbf{p}|^2p^4} - \frac{3k^4}{196|\mathbf{k}-\mathbf{p}|^2p^2} - \frac{k^2p^2}{49|\mathbf{k}-\mathbf{p}|^4} \\
& - \frac{15k^2|\mathbf{k}-\mathbf{p}|^2}{392p^4} + \frac{3k^4}{98|\mathbf{k}-\mathbf{p}|^4} - \frac{3k^2}{196|\mathbf{k}-\mathbf{p}|^2} - \frac{11k^4}{784p^4} + \frac{29k^2}{392p^2} + \frac{p^4}{196|\mathbf{k}-\mathbf{p}|^4} \\
& + \frac{3p^2}{196|\mathbf{k}-\mathbf{p}|^2} - \frac{15|\mathbf{k}-\mathbf{p}|^2}{392p^2} + \frac{25|\mathbf{k}-\mathbf{p}|^4}{784p^4} - \frac{11}{784},
\end{aligned} \tag{3.0.46}
$$

$$
\begin{aligned}
K_{\theta\theta}^{f_{\mathbf{k}-\mathbf{p}}f_p}(\mathbf{p},\mathbf{k}-\mathbf{p}) =& \frac{k^8}{98|\mathbf{k}-\mathbf{p}|^4p^4} - \frac{k^6}{196|\mathbf{k}-\mathbf{p}|^4p^2} - \frac{k^6}{196|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{37k^4}{392|\mathbf{k}-\mathbf{p}|^2p^2} + \frac{13k^2p^2}{196|\mathbf{k}-\mathbf{p}|^4} \\
&+ \frac{13k^2|\mathbf{k}-\mathbf{p}|^2}{196p^4} - \frac{9k^4}{196|\mathbf{k}-\mathbf{p}|^4} - \frac{13k^2}{196|\mathbf{k}-\mathbf{p}|^2} - \frac{9k^4}{196p^4} - \frac{13k^2}{196p^2} - \frac{9p^2}{392|\mathbf{k}-\mathbf{p}|^2} \\
&- \frac{5p^4}{196|\mathbf{k}-\mathbf{p}|^4} - \frac{9|\mathbf{k}-\mathbf{p}|^2}{392p^2} - \frac{5|\mathbf{k}-\mathbf{p}|^4}{196p^4} + \frac{19}{196},
\end{aligned} \tag{3.0.47}
$$

$$
\begin{aligned}
K_{\theta\theta}^{f_{\mathbf{p}^2}}(\mathbf{p},\mathbf{k}-\mathbf{p}) =& \frac{k^8}{196|\mathbf{k}-\mathbf{p}|^4p^4} + \frac{3k^6}{196|\mathbf{k}-\mathbf{p}|^4p^2} - \frac{k^6}{49|\mathbf{k}-\mathbf{p}|^2p^4} - \frac{3k^4}{196|\mathbf{k}-\mathbf{p}|^2p^2} - \frac{15k^2p^2}{392|\mathbf{k}-\mathbf{p}|^4} \\
&- \frac{k^2|\mathbf{k}-\mathbf{p}|^2}{49p^4} - \frac{11k^4}{784|\mathbf{k}-\mathbf{p}|^4} + \frac{29k^2}{392|\mathbf{k}-\mathbf{p}|^2} + \frac{3k^4}{98p^4} - \frac{3k^2}{196p^2} + \frac{25p^4}{784|\mathbf{k}-\mathbf{p}|^4} \\
&- \frac{15p^2}{392|\mathbf{k}-\mathbf{p}|^2} + \frac{3|\mathbf{k}-\mathbf{p}|^2}{196p^2} + \frac{|\mathbf{k}-\mathbf{p}|^4}{196p^4} - \frac{11}{784}.
\end{aligned} \tag{3.0.48}
$$

In the second equality of eq. (3.0.45) we have split $\left[G_2^{\mathbf{f_k}}(\mathbf{p},\mathbf{k}-\mathbf{p})\right]^2$ in three pieces, each of them proportional to the growth rate evaluated at different wavenumbers. Notice that under the interchange $\mathbf{p} \to \mathbf{k}-\mathbf{p}$, the above kernels satisfies $K_{\theta\theta}^{f_{(\mathbf{k}-\mathbf{p})^2}}(\mathbf{k}-\mathbf{p},\mathbf{p}) = K_{\theta\theta}^{f_{p^2}}(\mathbf{p},\mathbf{k}-\mathbf{p})$, so we can rewrite the eq. (3.0.45) as

$$
P_{cb,\theta\theta}^{22}(k) = 4\int_{\mathbf{p}} K_{\theta\theta}^{f_{\mathbf{p}^2}}(\mathbf{p},\mathbf{k}-\mathbf{p})P_{cb,\theta\theta}^{L}(p)P_L(|\mathbf{k}-\mathbf{p}|) + 2\int_{\mathbf{p}} K_{\theta\theta}^{f_{\mathbf{k}-\mathbf{p}}f_p}(\mathbf{p},\mathbf{k}-\mathbf{p})P_{cb,\delta\theta}^{L}(p)P_{cb,\delta\theta}^{L}(|\mathbf{k}-\mathbf{p}|), \tag{3.0.49}
$$

where

$$
K_{\theta\theta}^{f_{\mathbf{p}^2}}(\mathbf{p},\mathbf{k}-\mathbf{p}) \equiv \sum_{n_1,n_2=-2}^{2} f_{22,\theta\theta}^{f_{\mathbf{p}^2}}(n_1,n_2)k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2}, \tag{3.0.50}
$$

$$
f_{22,\theta\theta}^{f_{\mathbf{p}^2}}(n_1,n_2) = \begin{array}{cl}
\begin{array}{ccccc} n_2=-2 & -1 & 0 & 1 & 2 \end{array} & \\
\begin{pmatrix}
\frac{1}{196} & -\frac{1}{49} & \frac{3}{98} & -\frac{1}{49} & \frac{1}{196} \\
\frac{3}{196} & -\frac{3}{196} & -\frac{3}{196} & \frac{3}{196} & 0 \\
-\frac{11}{784} & \frac{29}{392} & -\frac{11}{784} & 0 & 0 \\
-\frac{15}{392} & -\frac{15}{392} & 0 & 0 & 0 \\
\frac{25}{784} & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{array}{c} n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}
\end{array}, \tag{3.0.51}
$$

and

$$
K_{\theta\theta}^{f_{\mathbf{k}-\mathbf{p}}f_p}(\mathbf{p},\mathbf{k}-\mathbf{p}) \equiv \sum_{n_1,n_2=-2}^{2} f_{22,\theta\theta}^{f_{\mathbf{k}-\mathbf{p}}f_{\mathbf{p}}}(n_1,n_2)k^{-2(n_1+n_2)}p^{2n_1}|\mathbf{k}-\mathbf{p}|^{2n_2}, \tag{3.0.52}
$$

$$
f_{22,\theta\theta}^{f_{\mathbf{k}-\mathbf{p}}f_{\mathbf{p}}}(n_1,n_2) = \begin{array}{cl}
\begin{array}{ccccc} n_2=-2 & -1 & 0 & 1 & 2 \end{array} & \\
\begin{pmatrix}
\frac{1}{98} & -\frac{1}{196} & -\frac{9}{196} & \frac{13}{196} & -\frac{5}{196} \\
-\frac{1}{196} & \frac{37}{392} & -\frac{13}{196} & -\frac{9}{392} & 0 \\
-\frac{9}{196} & -\frac{13}{196} & \frac{19}{196} & 0 & 0 \\
\frac{13}{196} & -\frac{9}{392} & 0 & 0 & 0 \\
-\frac{5}{196} & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{array}{c} n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}
\end{array}, \tag{3.0.53}
$$

where we have used $P^L_{cb,\delta\theta}(k) = (f(k)/f_0)\, P_L(k)$ and $P^L_{cb,\theta\theta}(k) = (f(k)/f_0)^2\, P_L(k)$. On the other hand, similar to the expansion carried out in eq. (3.0.34), we approximate

$$\bar{P}^L_{cb,\theta\theta}(k) = \sum_{m=-N/2}^{N/2} c_m^{ff} k^{\nu+i\eta_m}, \tag{3.0.54}$$

where the coefficients c_m^{ff} are computed in the same way as c_m, but changing $P_L(k) \to (f(k)/f_0)^2\, P_L(k)$. Then, using the above equations we can find the approximation for $P^{22}_{cb,\theta\theta}(k)$,

$$\bar{P}^{22}_{cb,\theta\theta}(k) = 2k^3 \sum_{m_1,m_2} c^{ff}_{m_1} k^{-2\nu_1} M^{f_{\mathbf{p}}^2}_{22,\theta\theta}(\nu_1,\nu_2) c_{m_2} k^{-2\nu_2} + k^3 \sum_{m_1,m_2} c^f_{m_1} k^{-2\nu_1} M^{f_{\mathbf{k}-\mathbf{p}} f_{\mathbf{p}}}_{22,\theta\theta}(\nu_1,\nu_2) c^f_{m_2} k^{-2\nu_2}, \tag{3.0.55}$$

with

$$\begin{aligned} M^{f_{\mathbf{p}}^2}_{22,\theta\theta}(\nu_1,\nu_2) = &\Big[98\nu_1^3\nu_2 + 7\nu_1^2(2\nu_2(7\nu_2-8)+1) - \nu_1(2\nu_2(7\nu_2+17)+53) - 12(1-2\nu_2)^2\Big] \\ &\times \frac{2\nu_{12}-3}{196\nu_1(\nu_1+1)\nu_2(\nu_2+1)(2\nu_2-1)} I(\nu_1,\nu_2), \end{aligned} \tag{3.0.56}$$

and

$$\begin{aligned} M^{f_{\mathbf{k}-\mathbf{p}} f_{\mathbf{p}}}_{22,\theta\theta}(\nu_1,\nu_2) = &\Big[7\nu_1^2(7\nu_2+3) + \nu_1(7\nu_2(7\nu_2-1)-10) + \nu_2(21\nu_2-10) - 37\Big] \\ &\times \frac{2\nu_{12}-3}{98\nu_1(\nu_1+1)\nu_2(\nu_2+1)} I(\nu_1,\nu_2). \end{aligned} \tag{3.0.57}$$

Finally, we compute the contribution $P^{13}_{cb,\theta\theta}(k)$ to the one-loop velocity power spectrum

$$\begin{aligned} P^{13}_{cb,\theta\theta}(k) &= 6 \int_{\mathbf{p}} G_3^{\mathbf{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p}) \frac{f(k)}{f_0} P_L(k) P_L(p) \\ &= 6 \left(\frac{f(k)}{f_0}\right)^2 P_L(k) \int_{\mathbf{p}} K^{f_{\mathbf{k}}}_{\theta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p}) P_L(p) + 6\, \frac{f(k)}{f_0} P_L(k) \int_{\mathbf{p}} K^{f_{\mathbf{p}}}_{\theta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p}) P^L_{\delta\theta}(p). \end{aligned} \tag{3.0.58}$$

Observe that have decomposed $G_3^{\mathbf{fk}}(\mathbf{k},-\mathbf{p},\mathbf{p})$ in two pieces, one of them proportional to $f(k)/f_0$ and the other proportional to $f(p)/f_0$, where

$$\begin{aligned} K^{f_{\mathbf{k}}}_{\theta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p}) = &-\frac{k^6}{252|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{k^4}{63|\mathbf{k}-\mathbf{p}|^2p^2} + \frac{19k^2|\mathbf{k}-\mathbf{p}|^2}{168p^4} - \frac{p^4}{252k^2|\mathbf{k}-\mathbf{p}|^2} - \frac{19|\mathbf{k}-\mathbf{p}|^4}{504k^2p^2} \\ &+ \frac{|\mathbf{k}-\mathbf{p}|^6}{72k^2p^4} - \frac{k^2}{42|\mathbf{k}-\mathbf{p}|^2} + \frac{5|\mathbf{k}-\mathbf{p}|^2}{168k^2} - \frac{11k^4}{252p^4} - \frac{41k^2}{504p^2} - \frac{p^2}{504k^2} + \frac{p^2}{63|\mathbf{k}-\mathbf{p}|^2} \\ &+ \frac{13|\mathbf{k}-\mathbf{p}|^2}{126p^2} - \frac{5|\mathbf{k}-\mathbf{p}|^4}{63p^4} - \frac{5}{126} \\ \equiv &\sum_{n_1=-2}^{2} \sum_{n_2=-1}^{3} f^{f_{\mathbf{k}}}_{13,\theta\theta}(n_1,n_2) k^{-2(n_1+n_2)} p^{2n_1} |\mathbf{k}-\mathbf{p}|^{2n_2}, \end{aligned} \tag{3.0.59}$$

$$\begin{aligned} K^{f_{\mathbf{p}}}_{\theta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p}) = &-\frac{k^6}{126|\mathbf{k}-\mathbf{p}|^2p^4} + \frac{k^4}{72|\mathbf{k}-\mathbf{p}|^2p^2} + \frac{k^2|\mathbf{k}-\mathbf{p}|^2}{168p^4} + \frac{5p^4}{504k^2|\mathbf{k}-\mathbf{p}|^2} - \frac{5|\mathbf{k}-\mathbf{p}|^4}{126k^2p^2} \\ &+ \frac{5|\mathbf{k}-\mathbf{p}|^6}{504k^2p^4} + \frac{k^2}{168|\mathbf{k}-\mathbf{p}|^2} + \frac{5|\mathbf{k}-\mathbf{p}|^2}{84k^2} + \frac{k^4}{72p^4} + \frac{k^2}{252p^2} - \frac{5p^2}{126k^2} - \frac{11p^2}{504|\mathbf{k}-\mathbf{p}|^2} \\ &+ \frac{11|\mathbf{k}-\mathbf{p}|^2}{504p^2} - \frac{11|\mathbf{k}-\mathbf{p}|^4}{504p^4} + \frac{11}{504} \\ = &K^{f_{\mathbf{p}}}_{\delta\theta}(\mathbf{k},-\mathbf{p},\mathbf{p}) \\ = &\sum_{n_1=-2}^{2} \sum_{n_2=-1}^{3} f^{f_{\mathbf{p}}}_{13,\delta\theta}(n_1,n_2) k^{-2(n_1+n_2)} p^{2n_1} |\mathbf{k}-\mathbf{p}|^{2n_2}, \end{aligned} \tag{3.0.60}$$

with

$$
f^{f_{\mathbf{k}}}_{13,\theta\theta}(n_1,n_2) = \begin{array}{c c}
\begin{array}{ccccc} n_2=-1 & 0 & 1 & 2 & 3 \end{array} & \\
\begin{pmatrix}
-\frac{1}{252} & -\frac{11}{252} & \frac{19}{168} & -\frac{5}{63} & \frac{1}{72} \\
\frac{1}{63} & -\frac{41}{504} & \frac{13}{126} & -\frac{19}{504} & 0 \\
-\frac{1}{42} & -\frac{5}{126} & \frac{5}{168} & 0 & 0 \\
\frac{1}{63} & -\frac{1}{504} & 0 & 0 & 0 \\
-\frac{1}{252} & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{array}{c} n_1=-2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}
\end{array}, \tag{3.0.61}
$$

while $f^{f_{\mathbf{P}}}_{13,\delta\theta}(n_1,n_2)$ is given by eq. (3.0.41). Then, the approximation for $P^{13}_{cb,\theta\theta}(k)$ gives

$$
\bar{P}^{13}_{\theta\theta}(k) = k^3\frac{f(k)}{f_0}P_L(k)\left(\frac{f(k)}{f_0}\sum_{m1} c_{m_1}k^{-2\nu_1}M^{f_{\mathbf{k}}}_{13,\theta\theta}(\nu_1) + \sum_{m1} c^f_{m_1}k^{-2\nu_1}M^{f_{\mathbf{P}}}_{13,\delta\theta}(\nu_1)\right), \tag{3.0.62}
$$

where

$$
M^{f_{\mathbf{k}}}_{13,\theta\theta}(\nu_1) = -\frac{\tan(\nu_1\pi)}{14\pi(\nu_1+1)\nu_1(\nu_1-1)(\nu_1-2)(\nu_1-3)}, \tag{3.0.63}
$$

while $M^{f_{\mathbf{P}}}_{13,\delta\theta}(\nu_1)$ is given by the eq. (3.0.43).

We have reduced the loop integrals to simple matrix multiplications. Furthermore, the matrices M_{22} and vectors M_{13} depend on the FFTLog parameters rather than the cosmological parameters, which significantly impacts the computational time of an Markov chain Monte Carlo (MCMC) exploration.

4 One-loop power spectrum of biased tracers

It is well known that the spatial clustering pattern of observable tracers and CDM-baryons is not necessarily the same [13]. In general, there is a bias between the density contrast of tracers $\delta(\mathbf{x})$ and the bulk of CDM-baryons $\delta_{cb}(\mathbf{x})$, which depends on the tracers and their redshift. To model the bias, we follow the biasing scheme presented in ref. [14] and recently generalized in ref. [15] to cosmologies containing additional scales. In this scheme, the large scale tracer velocities are assumed to be unbiased, so they follow the same geodesics of the CDM-baryons fluid, while tracer density contrast is biased by

$$
\delta(\boldsymbol{x}) = c_\delta\delta_{cb} + \frac{1}{2}c_{\delta^2}\delta^2_{cb} + c_{s^2}s^2 + \frac{1}{6}c_{\delta^3}\delta^3_{cb} + \frac{1}{2}c_{\delta s^2}\delta_{cb}s^2 + c_\psi\psi + c_{st}st + \frac{1}{6}c_{s^3}s^3 + \text{stochastic terms}, \tag{4.1}
$$

where $s^2 = s_{ij}s_{ij}$, $st = s_{ij}t_{ij}$ and $s^3 = s_{ij}s_{jk}s_{ki}$, with

$$
s_{ij}(\boldsymbol{k}) = \left(\frac{k_ik_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\delta_{cb}(\boldsymbol{k}), \qquad t_{ij}(\boldsymbol{k}) = \left(\frac{k_ik_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\eta(\boldsymbol{k}), \tag{4.2}
$$

and

$$
\eta(\boldsymbol{k}) = \theta_{cb}(\boldsymbol{k}) - \frac{f(k)}{f_0}\delta_{cb}(\boldsymbol{k}), \tag{4.3}
$$

their Fourier representations. Notice that s^2 and s^3 are second-order and third-order operators, respectively. While st is a third-order operator since η vanishes at linear order via eq. (1.2). Furthermore, the operator ψ is defined as

$$
\psi(\boldsymbol{k}) = \eta(\boldsymbol{k}) - \frac{f(k)}{f_0}\left(\frac{2}{7}s^2(\boldsymbol{k}) - \frac{4}{21}\delta^2_{cb}(\boldsymbol{k})\right), \tag{4.4}
$$

which is a third-order operator for EdS kernels since $\eta^{(2)} = \frac{2}{7}s^2 - \frac{4}{21}\delta^2$ and $f(k) = f_0$. However, for ΛCDM kernels the expression for $\eta^{(2)}$ does not holds, causing the operator ψ to receive second-order contributions. In the case of massless neutrinos, these second-order contributions can be absorbed into the s^2 and δ^2 operators since they are degenerate. So, in this case, ψ can be conceived as a third-order operator [16]. However, when considering massive neutrinos, the last statement is not true, but since its influence is fairly small, we can treat ψ as a third-order operator.

From the bias expansion (4.1) and after renormalization, one finds that the one-loop power spectrum for biased tracer is

$$
\begin{aligned}
P_{\delta\delta}(k) &= b_1^2 P^{\text{1-loop}}_{cb,\delta\delta}(k) + 2b_1 b_2 P_{b_1 b_2}(k) + 2b_1 b_{s^2} P_{b_1 b_{s^2}}(k) + b_2^2 P_{b_2^2}(k) \\
&\quad + 2b_2 b_{s^2} P_{b_2 b_{s^2}}(k) + b_{s^2}^2 P_{b_{s^2}^2}(k) + 2b_1 b_{3nl}\sigma_3^2(k) P_L(k),
\end{aligned} \tag{4.5}
$$

$$
P_{\delta\theta}(k) = b_1 P^{\text{1-loop}}_{cb,\delta\theta}(k) + b_2 P_{b_2,\theta}(k) + b_{s^2} P_{b_{s^2},\theta}(k) + b_{3nl}\sigma_3^2(k) P^L_{cb,\delta\theta}(k), \tag{4.6}
$$

$$
P_{\theta\theta}(k) = P^{\text{1-loop}}_{cb,\theta\theta}(k), \tag{4.7}
$$

where b_1, b_2, b_{s^2} and b_{3nl} are the bias parameters, while the quantities $P^{\text{1-loop}}_{cb,\delta\delta}(k)$, $P^{\text{1-loop}}_{cb,\delta\theta}(k)$ and $P^{\text{1-loop}}_{cb,\theta\theta}(k)$ are given by eqs. (3.3)–(3.5). The other contributions are

$$
P_{b_1 b_2}(k) = \int_{\mathbf{p}} F_2^{\mathbf{fk}}(\mathbf{p}, \mathbf{k}-\mathbf{p}) P_L(p) P_L(|\mathbf{k}-\mathbf{p}|), \qquad (-3 < \nu < -1/2) \tag{4.8}
$$

$$
P_{b_1 b_{s^2}}(k) = \int_{\mathbf{p}} F_2^{\mathbf{fk}}(\mathbf{p}, \mathbf{k}-\mathbf{p}) S_2(\mathbf{p}, \mathbf{k}-\mathbf{p}) P_L(p) P_L(|\mathbf{k}-\mathbf{p}|), \qquad (-3 < \nu < 1/2) \tag{4.9}
$$

$$
P_{b_2^2}(k) = \frac{1}{2}\int_{\mathbf{p}} P_L(p)\big[P_L(|\mathbf{k}-\mathbf{p}|) - P_L(p)\big], \qquad (-3 < \nu) \tag{4.10}
$$

$$
P_{b_2 b_{s^2}}(k) = \frac{1}{2}\int_{\mathbf{p}} P_L(p)\left[P_L(|\mathbf{k}-\mathbf{p}|) S_2(\mathbf{p}, \mathbf{k}-\mathbf{p}) - \frac{2}{3}P_L(p)\right], \qquad (-3 < \nu) \tag{4.11}
$$

$$
P_{b_{s^2}^2}(k) = \frac{1}{2}\int_{\mathbf{p}} P_L(p)\left[P_L(|\mathbf{k}-\mathbf{p}|)\, [S_2(\mathbf{p}, \mathbf{k}-\mathbf{p})]^2 - \frac{4}{9}P_L(p)\right], \qquad (-3 < \nu) \tag{4.12}
$$

$$
P_{b_2,\theta}(k) = \int_{\mathbf{p}} G_2^{\mathbf{fk}}(\mathbf{p}, \mathbf{k}-\mathbf{p}) P_L(p) P_L(|\mathbf{k}-\mathbf{p}|), \qquad (-3 < \nu < -1/2) \tag{4.13}
$$

$$
P_{b_{s^2},\theta}(k) = \int_{\mathbf{p}} G_2^{\mathbf{fk}}(\mathbf{p}, \mathbf{k}-\mathbf{p}) S_2(\mathbf{p}, \mathbf{k}-\mathbf{p}) P_L(p) P_L(|\mathbf{k}-\mathbf{p}|), \qquad (-3 < \nu < 1/2) \tag{4.14}
$$

Additionally, the function $\sigma_3^2(k)$ is given by,

$$
\sigma_3^2(k) = \frac{105}{16}\int_{\mathbf{p}} P_L(p)\left[S_2(\mathbf{p}, \mathbf{k}-\mathbf{p})\left(\frac{2}{7}S_2(-\mathbf{p}, \mathbf{k}) - \frac{4}{21}\right) + \frac{8}{63}\right], \tag{4.15}
$$

where

$$
S_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} - \frac{1}{3}. \tag{4.16}
$$

Following similar steps to those performed for the previous case of one-loop matter power spectrum, we find

$$
\bar{P}(k) = k^3 \sum_{m_1, m_2} c_{m_1} k^{-2\nu_1}\, M_P(\nu_1, \nu_2)\, c_{m_2} k^{-2\nu_2}, \tag{4.17}
$$

where P refers to the previous contributions $P_{b_1b_2}$, $P_{b_1b_{s^2}}$, $P_{b_2^2}$, $P_{b_2b_{s^2}}$ and $P_{b_{s^2}^2}$, while the matrices involved are

$$M_{P_{b_1b_2}}(\nu_1,\nu_2) = \frac{(2\nu_{12}-3)(7\nu_{12}-4)}{28\nu_1\nu_2}I(\nu_1,\nu_2), \tag{4.18}$$

$$M_{P_{b_1b_{s^2}}}(\nu_1,\nu_2) = \frac{(2\nu_{12}-3)\Big[14\nu_1^2(2\nu_2-1)+\nu_1\big(4\nu_2(7\nu_2-11)-3\big)-\nu_2(14\nu_2+3)+2\Big]}{168\nu_1(\nu_1+1)\nu_2(\nu_2+1)}I(\nu_1,\nu_2), \tag{4.19}$$

$$M_{P_{b_2^2}}(\nu_1,\nu_2) = \frac{1}{2}I(\nu_1,\nu_2) \tag{4.20}$$

$$M_{P_{b_2b_{s^2}}}(\nu_1,\nu_2) = \frac{(2\nu_1-3)(2\nu_2-3)}{12\nu_1\nu_2}I(\nu_1,\nu_2), \tag{4.21}$$

$$M_{P_{b_{s^2}^2}}(\nu_1,\nu_2) = \frac{\Big[4\Big((3+2(\nu_1-2)\nu_1)\nu_2^2+(17-4\nu_1)\nu_1\nu_2+3(\nu_1-5)\nu_1\Big)-60\nu_2+63\Big]}{36\nu_1(\nu_1+1)\nu_2(\nu_2+1)}I(\nu_1,\nu_2), \tag{4.22}$$

Notice that for the contributions (4.10)$-$(4.12), we have to subtract the terms $P_{b_2^2}(0)$, $P_{b_2b_{s^2}}(0)$, $P_{b_{s^2}^2}(0)$, respectively.

The FFTLog approximations for the eqs.() and () can be expressed as

$$\bar{P}(k) = k^3\sum_{m_1,m_2} c^f_{m_1}k^{-2\nu_1}\,M_P(\nu_1,\nu_2)\,c_{m_2}k^{-2\nu_2}, \tag{4.23}$$

where P stand for $P_{b_{s^2},\theta}$ and $P_{b_2,\theta}$, with matrices given by

$$M_{P_{b_{s^2},\theta}}(\nu_1,\nu_2) = \frac{(2\nu_{12}-3)\Big[\nu_1\Big(14\nu_1(2\nu_2-1)-30\nu_2+39\Big)-10\nu_2-19\Big]}{84\nu_1(\nu_1+1)\nu_2(\nu_2+1)}I(\nu_1,\nu_2), \tag{4.24}$$

$$M_{P_{b_2,\theta}}(\nu_1,\nu_2) = \frac{(7\nu_1-4)(2\nu_{12}-3)}{14\nu_1\nu_2}I(\nu_1,\nu_2). \tag{4.25}$$

Finally, the approximation for the function $\sigma_3^2(k)$ is
where

$$M_{\sigma_3^2}(\nu_1) = \frac{45\tan(\pi\nu 1)}{128\pi(\nu_1-3)(\nu_1-2)(\nu_1-1)\nu_1(\nu_1+1)}. \tag{4.26}$$

5 Performance and accuracy of the FFTLog formalism

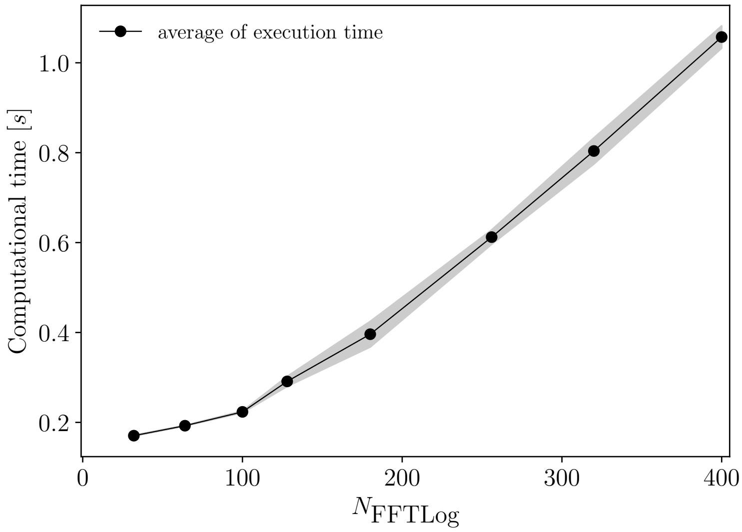

Figure 1: Performance

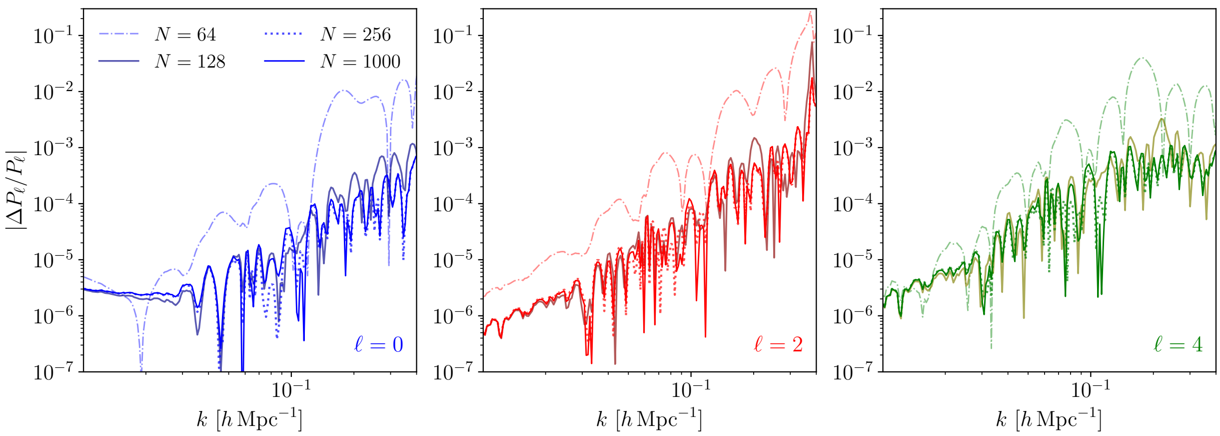

Figure 2: Performance

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